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# Large deformation contact of a rubber notch with a rigid wedge

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**Abstract**

The contact problem of a rubber notch with a rigid wedge is analyzed based on large strain theory. The basic equations of deformation field near the notch corner are derived and solved. Analytical solution is obtained for expanding sector while the numerical solution is given for two shrinking sectors. The singularity exponent of stress and strain field is expressed by the angle of rigid wedge and the constitutive parameter of the material. A special interesting case is that a half rubber plane contacts with a rigid wedge, for which the completely analytical solutions are obtained for both expanding sector and shrinking sectors. The analysis of this paper is also valid for the contact problem of a rubber wedge with a rigid wedge. © 2000 Elsevier Science Ltd. All rights reserved.

*Keywords:* Vertex contact; Large strain; Rubber notch

## 1. Introduction

The first solution to contact problem was given by Hertz (1881) for two spherical bodies, based on linear elastic theory. The general contact problem of two bodies with smooth surface can be solved by means of integral equations or calculated by finite element method.

When the contact surfaces contain vertex, the problem cannot be solved in the framework of infinitesimal strain theory because it will give zero contact area. Therefore, the vertex contact problem must be solved by large strain theory. It should be noted that in nonlinear elasticity, the summation principle is not valid, therefore, the solution of a concentrated force problem cannot be used to solve the contact problem. That is why the vertex contact problem has not been solved until now.

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In the present paper, the vertex contact problem is solved based on the elastic law given by Gao (1997), the sector division method given by Gao (1990) is also used.

## 2. Basic formulae

Consider a three-dimensional domain of rubber material. $\boldsymbol{P}$ and $\boldsymbol{Q}$ denote the position vectors of a point before and after deformation respectively. $x^i$ ($i = 1, 2, 3$) is the Lagrangian coordinate. Two set of local triads can be defined as

$$\boldsymbol{P}_i = \frac{\partial \boldsymbol{P}}{\partial x^i}, \quad \boldsymbol{Q}_i = \frac{\partial \boldsymbol{Q}}{\partial x^i} \tag{1}$$

Three independent invariants based on $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$ can be introduced

$$\begin{cases} I = (\boldsymbol{P}^i \cdot \boldsymbol{P}^j)(\boldsymbol{Q}_i \cdot \boldsymbol{Q}_j), & I_{-1} = (\boldsymbol{P}_i \cdot \boldsymbol{P}_j)(\boldsymbol{Q}^i \cdot \boldsymbol{Q}^j) \\ J = \dfrac{V_Q}{V_P}, & V_* = (*_1, *_2, *_3) \end{cases} . \tag{2}$$

in which summation rule is implied; $\boldsymbol{P}^i$ and $\boldsymbol{Q}^i$ are the conjugates of $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$, respectively, i.e. $\boldsymbol{P}_i \cdot \boldsymbol{P}^j = \delta_i^j$, $\boldsymbol{Q}_i \cdot \boldsymbol{Q}^j = \delta_i^j$; $(*_1, *_2, *_3)$ denotes the mixed product of vectors $*_1$, $*_2$ and $*_3$. A strain energy function per undeformed unit volume is proposed by Gao (1997),

$$U = a\left(I^n + I_{-1}^n\right) \tag{3}$$

This strain energy was successfully used in solving a series of typical problems, (Gao (1997, 1998, 1999a)).

Mooney (1940) and Rivlin (1949) pointed out that the strain energy function can be expanded as an infinite series of $I$ and $I_2$,

$$U = \sum_{m,n=0}^{\infty} C_{mn}(I-3)^m (I_2 - 3)^n \tag{4}$$

where

$$I_2 = (\boldsymbol{P}^i \cdot \boldsymbol{P}^j)(\boldsymbol{Q}_j \cdot \boldsymbol{Q}_k)(\boldsymbol{P}^k \cdot \boldsymbol{P}^l)(\boldsymbol{Q}_l \cdot \boldsymbol{Q}_i) \tag{5}$$

Because $I_{-1}$ can be expressed as

$$I_{-1} = \frac{1}{2J^2}\left(I^2 - I_2\right) \tag{6}$$

then Eq. (3) can be written as

$$U = a\left[I^n + \frac{1}{2^n J^{2n}}\left(I^2 - I_2\right)^n\right] \tag{7}$$

For incompressible material, $J \equiv 1$, evidently, form (3) belongs to the class of function (4), but for compressible material (3) cannot be contained in form (4). The original purpose of form (3) is to reduce the energy function as simple as possible but it must reflect the basic nature of solid materials. Actually the first term of (3) mainly expresses the resistance of material to large tension while the second term of

(3) mainly deputes to the resistance to tremendous compression. Therefore with these two terms, the material natures are complete. The detailed discussion on the material behavior governed by (3) can be found in Gao (1999b).

From strain energy $U$, the Cauchy stress can be obtained,

$$\tau = J^{-1}\frac{\partial U}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i \tag{8}$$

where $\otimes$ is the dyadic symbol.

The following relations are important,

$$\begin{cases} \dfrac{\partial I}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i = 2d, \qquad \dfrac{\partial I_{-1}}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i = -2d^{-1} \\ \dfrac{\partial J}{\partial \boldsymbol{Q}_i} \otimes \boldsymbol{Q}_i = J \cdot \boldsymbol{E} \end{cases} \tag{9}$$

in which

$$\begin{cases} \boldsymbol{d} = (\boldsymbol{P}^i \cdot \boldsymbol{P}^j)\boldsymbol{Q}_i \otimes \boldsymbol{Q}_j, \qquad \boldsymbol{d}^{-1} = (\boldsymbol{P}_i \cdot \boldsymbol{P}_j)\boldsymbol{Q}^i \otimes \boldsymbol{Q}^j \\ \boldsymbol{E} = \boldsymbol{P}_i \otimes \boldsymbol{P}^i = \boldsymbol{Q}_i \otimes \boldsymbol{Q}^i \end{cases} \tag{10}$$

Eqs. (3), (8)–(9) can be combined to give

$$\tau = 2naJ^{-1}\left(I^{n-1}\boldsymbol{d} - I_{-1}^{n-1}\boldsymbol{d}^{-1}\right) \tag{11}$$

The equilibrium equation can be written as

$$\frac{\partial}{\partial x^i}\left(V_Q \tau \cdot \boldsymbol{Q}^i\right) = 0 \tag{12}$$

According to Eqs. (2) and (8), the equivalent form of (12) is

$$\frac{\partial}{\partial x^i}\left(V_P \frac{\partial U}{\partial \boldsymbol{Q}_i}\right) = 0 \tag{13}$$

The traction free boundary condition is

$$\tau \cdot \boldsymbol{B} = 0 \tag{14}$$

where, $\boldsymbol{B}$ is the unit normal vector of the deformed boundary, i.e.

$$\boldsymbol{B} = \frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}b} = \boldsymbol{Q}_i \cdot \frac{\mathrm{d}x^i}{\mathrm{d}b} \tag{15}$$

in which $b$ is the distance to the boundary from outside. Eqs. (8) and (15) can be used to rewrite Eq. (14) as

$$\frac{\partial U}{\partial \boldsymbol{Q}_i} \cdot \frac{\partial b}{\partial x^i} = 0 \tag{16}$$

## 3. Deformation pattern

Shown in Fig. 1(a) and (b) are the cross sections of a notched rubber with a rigid wedge before and after deformation respectively. The angle of wedge is $2A$ while the angle of rubber notch is $2B$.

For simplicity, only the symmetric loading case is considered here. In order to describe the deformation, the vicinity of notch corner is divided into different sectors. SH and SH′ are called shrinking sectors; before deformation they occupy almost the whole notch corner domain, but after deformation they become very narrow and located near the boundaries of the rigid wedge, as shown in Fig. 1. EX is called expanding sector; before deformation it is very narrow, but after deformation it becomes very wide and occupies almost the whole notch corner domain. Because of the different deformation feature, sectors SH (SH′) and EX must be described by different mapping functions. Two sets of Lagrangian coordinates are introduced such that $R, \Theta, Z$ are cylinder coordinates before deformation while $r, \theta, z$ are cylinder coordinates after deformation. The deformation is considered as plane strain case so $Z \equiv z$, and it is enough to consider the mapping functions from $(r, \theta)$ to $(R, \Theta)$. In shrinking sectors SH, the mapping functions are assumed as,

$$\begin{cases} R = r^{1+\beta} f(\xi), & \Theta = g(\xi) \\ \xi = r^{-\alpha}(\theta - \pi + A), & -\infty < \xi < 0 \end{cases}. \tag{17}$$

where $\alpha$, $\beta$ are positive exponents, $A$ is the half angle of the rigid wedge. The mapping functions for sector SH′ can be given similarly, but it is omitted.

In sector EX, the mapping functions are assumed as

$$R = r^{1-\delta} h(\theta), \quad \Theta = r^{\gamma} m(\theta) \quad |\theta| < \pi - A \tag{18}$$

in which $\delta$ and $\gamma$ are positive exponents.

Actually, there is no exact boundary between different sectors. Mapping functions (17) and (18) are convertible one to another at the fuzzy boundaries.

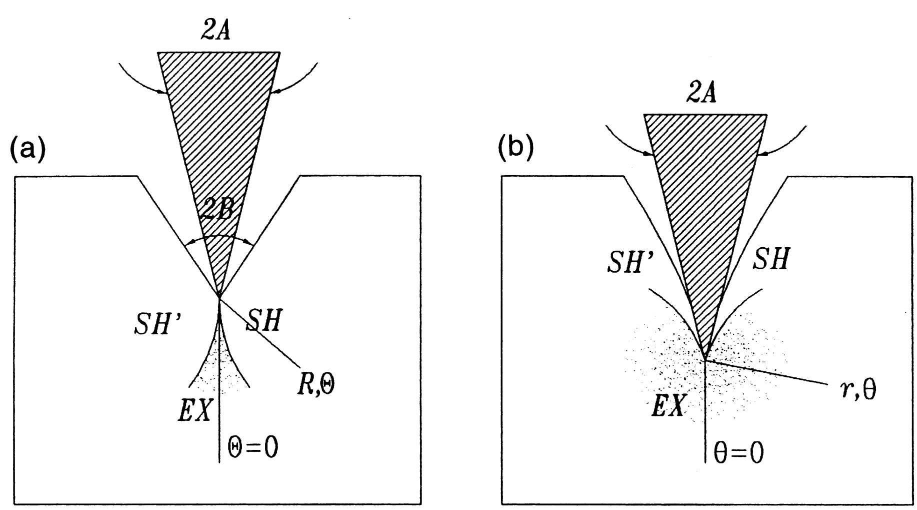


Fig. 1. A rubber notch with a rigid wedge (a) before deformation (b) after deformation.

## 4. Expanding sector EX

Let $\boldsymbol{e}_R$, $\boldsymbol{e}_\Theta$, $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ denote the unit vectors before and after deformation, in $(R, \Theta)$ and $(r, \theta)$ systems respectively, i.e.

$$\boldsymbol{e}_R = \frac{\partial \boldsymbol{P}}{\partial R} = \boldsymbol{P}_R, \quad \boldsymbol{e}_\Theta = \frac{1}{R}\frac{\partial \boldsymbol{P}}{\partial \Theta} = \frac{1}{R}\boldsymbol{P}_\Theta \tag{19}$$

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{Q}}{\partial r} = \boldsymbol{Q}_r, \quad \boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{Q}}{\partial \theta} = \frac{1}{r}\boldsymbol{Q}_\theta \tag{20}$$

then according to Eqs. (18) and (19), it follows that

$$\begin{cases} \boldsymbol{P}_r = \dfrac{\partial \boldsymbol{P}}{\partial r} = r^{-\delta}h\big[(1-\delta)\boldsymbol{e}_R + \gamma r^\gamma m \boldsymbol{e}_\Theta\big] \\ \boldsymbol{P}_\theta = \dfrac{\partial \boldsymbol{P}}{\partial \theta} = r^{1-\delta}(h'\boldsymbol{e}_R + r^\gamma h m' \boldsymbol{e}_\Theta) \end{cases} \tag{21}$$

Eq. (21) can be inverted to give

$$\begin{cases} \boldsymbol{P}^r = r^{\delta-\gamma}q^{-1}(r^\gamma h m' \boldsymbol{e}_R - h'\boldsymbol{e}_\Theta) \\ \boldsymbol{P}^\theta = r^{\delta-\gamma-1}q^{-1}h\big[-\gamma r^\gamma m \boldsymbol{e}_R + (1-\delta)\boldsymbol{e}_\Theta\big] \end{cases} \tag{22}$$

where

$$q = h\big[(1-\delta)hm' - \gamma h' m\big] \tag{23}$$

Using (2) and (19)–(22), it follows that,

$$\begin{cases} I = r^{2\delta-2\gamma}q^{-2}p, \quad I_{-1} = r^{-2\delta}p \\ J = r^{2\delta\gamma}q^{-1} \end{cases} \tag{24}$$

where

$$p = h'^{\,2} + (1-\delta)^2 h^2 \tag{25}$$

Eq. (24) shows that $q$ is a quantity that is inverse proportion to volume inflation. Using Eqs. (19)–(22) and (10), it follows that,

$$\boldsymbol{d} = r^{2\delta-2\gamma}q^{-2}\Big[h'^{\,2}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + (1-\delta)^2 h^2 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - (1-\delta)hh'(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r)\Big] \tag{26}$$

$$\boldsymbol{d}^{-1} = r^{-2\delta}\Big[h'^{\,2}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + (1-\delta)^2 h^2 \boldsymbol{e}_r \otimes \boldsymbol{e}_r + (1-\delta)hh'(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r)\Big] \tag{27}$$

In order to match the two terms in Eq. (11), it is required that,

$$\gamma = 2\delta \tag{28}$$

Then Eqs. (11) and (24)–(27) are combined to give

$$\begin{cases} V_Q\tau\cdot\boldsymbol{Q}^r = 2nar^{1-2n\delta}p^{n-1}q\Big\{\big[h'^{\,2}Y-(1-\delta)^2h^2\big]\boldsymbol{e}_r-(1-\delta)hh'(1+Y)\boldsymbol{e}_\theta\Big\} \\ V_Q\tau\cdot\boldsymbol{Q}^\theta = 2nar^{-2n\delta}p^{n-1}q\Big\{-(1-\delta)hh'(1+Y)\boldsymbol{e}_r+\big[(1-\delta)^2h^2Y-h'^{\,2}\big]\boldsymbol{e}_\theta\Big\} \end{cases} \tag{29}$$

in which $Y$ is given by $q$,

$$Y = q^{-2n} \tag{30}$$

Substituting Eq. (29) into (12), after a long manipulation it follows that

$$\left[h''+\frac{\delta}{1-\delta}\frac{h'^{\,2}}{h}+(1-\delta)h\right]\Delta-2n\delta(1-\delta)^3Yh^3-\frac{2n\delta Y}{1-\delta}\frac{h'^{\,4}}{h}-4\delta(1-\delta)nYhh'^{\,2}=0 \tag{31}$$

$$\begin{aligned}&\frac{1-\delta}{q}h^2wm''+\left(\frac{h'}{h}+\frac{1-\delta-\gamma}{q}hh'm'\right)w+2hh'\left\{\frac{n-1}{p}\big[h'^{\,2}-(1-\delta)^2Yh^2\big]+(1-\delta)\big[1-n\delta\right.\\ &\left.-(n-1)\delta Y\big]\right\}+h''\left\{2h'-\frac{\gamma mh}{q}w+\frac{2(n-1)}{p}h'\big[h'^{\,2}-(1-\delta)^2Yh^2\big]\right\}\\ &=0\end{aligned} \tag{32}$$

in which

$$\begin{cases}\Delta = \left[(2n+1)Y-\dfrac{1}{2n-1}\right]h'^{\,2}+(1-\delta)^2Y(1+Y)h^2 \\ w=(2n-1)(1-\delta)^2Yh^2+h'^{\,2}\end{cases} \tag{33}$$

Eqs. (31) and (32) can be solved numerically. The symmetric boundary conditions at $\theta=0$ are,

$$h'(0)=0,\quad h(0)=h_0 \tag{34}$$

$$m(0)=0,\quad m'(0)=m_1 \tag{35}$$

At $\theta=\pi-A$, the natural boundary conditions must be satisfied to connect with sector SH,

$$h(\pi-A)=0 \tag{36}$$

$$m(\pi-A)=\infty \tag{37}$$

So, there are three parameters $h_0, m$ and $\delta$ to be determined. $h_0$ can be considered as a free parameter to indicate the amplitude of the field. $m_1$ and $\delta$ can be adjusted to meet the condition (36) and (37) as well as the match conditions with shrinking sectors. However, the behavior of functions $h$ and $m$ is not easy to be revealed because there are two parameters $\delta$ and $m_1$. It is found by Gao (1998a) that for a wedge acted by a concentrated force, in sector EX,

$$q\equiv(2n-1)^{\frac{1}{2n}} \tag{38}$$

The result for a concentrated force problem hints us to test relation (38) for Eqs. (31) and (32).

Surprisingly, when (38) is used, Eqs. (31) and (32) become the same equation as,

$$\left[h'' + \frac{\delta}{1-\delta}\frac{h'^2}{h} + (1-\delta)h\right]\left[h'^2 + \frac{(1-\delta)^2}{2n-1}h^2\right] - \frac{\delta}{1-\delta}\frac{p^2}{h} = 0 \tag{39}$$

Therefore (38) is a solution of (31) and (32), then (31) and (32) are reduced to (38) and (39).

Further, (38) and (39) can be rewritten as

$$\left[h'' + (1-\delta)^2 h\right]\left[1 + 2(n-1)h'^2 p^{-1}\right] - 2(n-1)\delta(1-\delta)h = 0 \tag{40}$$

$$m' = \frac{1}{(1-\delta)h^2}\left[(2n-1)^{\frac{1}{2n}} + \gamma h h' m\right] \tag{41}$$

For Eq. (41), $m'(0)$ $(= m_1)$ is no longer a free parameter while it depends on $h_0$,

$$m'(0) = \frac{1}{(1-\delta)h_0^2}(2n-1)^{\frac{1}{2n}} \tag{42}$$

Therefore the free parameters are only $\delta$ and $h_0$. Evidently, $h_0$ has no relation with condition (36), so $\delta$ can be determined by (36) and the first of (34). The analytical solution of Eq. (40) under boundary conditions (34) and (36) can be found in Gao (1999a),

$$\begin{cases} \delta = \dfrac{ne}{(2n-1)(1-e)}\left\{\left[1 + \dfrac{1-e}{n^2 e}(2n-1)\right]^{1/2} - 1\right\} \\ e = \left(1 - \dfrac{2A}{\pi}\right)^2 \end{cases} \tag{43}$$

$$\begin{cases} h = h_0\left[1-(2n-1)\delta\right]^{\frac{\delta}{2}}\left[n-(2n-1)\delta-(n-1)\cos 2x\right]^{-\frac{\delta}{2}}\cos x \\ \theta = x - \dfrac{\delta}{\varepsilon(1-\delta)}\mathrm{arctg}(\varepsilon\cdot\cot x) + \dfrac{\pi}{2} - A \\ \varepsilon = \left[\dfrac{1-(2n-1)\delta}{(2n-1)(1-\delta)}\right]^{1/2}, \quad 0 < x < \dfrac{\pi}{2} \end{cases} \tag{44}$$

where $x$ is introduced as

$$tgx = \frac{h'}{(1-\delta)h} \tag{45}$$

$x$ can be considered as a parametric variable. Eq. (43) shows that the singular exponent $\delta$ only depends on exponent $n$ and angle $A$ but not on angle $B$.

A special case is $A = 0$, then the contact problem becomes a concentrated force problem, it follows that

$$\delta = \frac{1}{2n} \tag{46}$$

$$\begin{cases} h = h_0[1-(2n-1)\delta]^{\frac{\delta}{2}}\left(\frac{n}{2}\right)^{\frac{1-\delta}{2}}(\Omega+\cos\theta)^{\frac{1}{2}}\left[\Omega-\left(1-\frac{1}{n}\right)\cos\theta\right]^{\frac{1}{2}-\frac{1}{2n}} \\ \Omega = \left[1-\left(1-\frac{1}{n}\right)^2\sin^2\theta\right]^{\frac{1}{2}} \end{cases} \quad (47)$$

Eq. (46) is consistent with the result obtained by Gao (1998), but the analytical solution (47) has not been obtained there. When $h$ is obtained, function $m$ can calculated according to Eq. (41) and the first of conditions (35).

## 5. Shrinking sector SH

In sector SH, the mapping function (17) is adopted. For simplicity, we introduce the ($\eta$, $\xi$) coordinate as shown in Fig. 2,

$$\eta = r\left(1+\frac{\alpha}{2}\theta^{*^2}\right), \; \xi = r^{-\alpha}\theta^*, \quad \theta^* = \theta - \pi + A \quad (48)$$

in which $\theta^*$ is a very small variable. When high order terms of $\theta^*$ are neglected, ($\eta$, $\xi$) are orthogonal coordinates. The inverse expression of (48) is

$$r = \eta\left(1-\frac{\alpha}{2}\eta^{2\alpha}\xi^2\right), \; \theta^* = \eta^\alpha\xi \quad (49)$$

then (17) can be written as

$$R = \eta^{1+\beta}f(\xi), \; \Theta = g(\xi) \quad (50)$$

According to Eqs. (1), (19), (20), (49) and (50), it follows that,

$$\boldsymbol{Q}_\eta = \boldsymbol{e}_r + \eta^\alpha\alpha\xi\boldsymbol{e}_\theta, \; \boldsymbol{Q}_\xi = \eta^{1+\alpha}\left(\boldsymbol{e}_\theta - \eta^\alpha\alpha\xi\boldsymbol{e}_r\right) \quad (51)$$

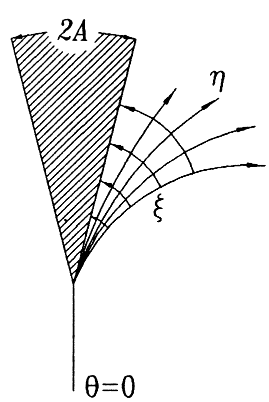


Fig. 2. The coordinate ($\eta$, $\xi$) in sector SH.

$$\boldsymbol{P}_\eta = (1+\beta)\eta^\beta f \boldsymbol{e}_R,\ \boldsymbol{P}_\xi = \eta^{1+\beta}\left(f'\boldsymbol{e}_R + fg'\boldsymbol{e}_\Theta\right) \tag{52}$$

From Eqs. (51) and (52) following is obtained,

$$\boldsymbol{Q}^\eta = \boldsymbol{e}_r + \eta^\alpha \alpha \xi \boldsymbol{e}_\theta,\ \boldsymbol{Q}^\xi = \eta^{-1-\alpha}\left(\boldsymbol{e}_\theta - \eta^\alpha \alpha \xi \boldsymbol{e}_r\right) \tag{53}$$

$$\boldsymbol{P}^\eta = \eta^{-\beta} v^{-1}\left(fg'\boldsymbol{e}_R - f'\boldsymbol{e}_\Theta\right),\ \boldsymbol{P}^\xi = \eta^{-1-\beta}(1+\beta) f v^{-1}\boldsymbol{e}_\Theta \tag{54}$$

where

$$v = (1+\beta) f^{\,2} g' \tag{55}$$

From Eqs. (2), (51)–(54), the invariants are obtained,

$$\begin{cases} I = \eta^{-2\beta} u v^{-2} & I_{-1} = \eta^{2\beta-2\alpha} u \\ J = \eta^{\alpha-2\beta} v^{-1} & u = f'^{\,2} + f^{\,2} g'^{\,2} \end{cases} \tag{56}$$

Eqs. (51)–(54) and (10) can be used to give

$$\boldsymbol{d} = \eta^{-2\beta} v^{-2}\Big[u\boldsymbol{Q}_\eta \otimes \boldsymbol{Q}_\eta - \eta^{-1}(1+\beta) f f'\left(\boldsymbol{Q}_\eta \otimes \boldsymbol{Q}_\xi + \boldsymbol{Q}_\xi \otimes \boldsymbol{Q}_\eta\right) + \eta^{-2}(1+\beta)^2 f^{\,2} \boldsymbol{Q}_\xi \otimes \boldsymbol{Q}_\xi\Big] \tag{57}$$

$$\boldsymbol{d}^{-1} = \eta^{2\beta}\left[(1+\beta)^2 f^{\,2} \boldsymbol{Q}^\eta \otimes \boldsymbol{Q}^\eta + \eta(1+\beta) f f'\left(\boldsymbol{Q}^\xi \otimes \boldsymbol{Q}^\eta + \boldsymbol{Q}^\eta \otimes \boldsymbol{Q}^\xi\right) + \eta^2 u \boldsymbol{Q}^\xi \otimes \boldsymbol{Q}^\xi\right] \tag{58}$$

In order to match the singularity of $\boldsymbol{d}$ and $\boldsymbol{d}^{-1}$ in Eq. (11), it is required that

$$\alpha = 2\beta \tag{59}$$

then only taking the dominant terms, Eqs. (11), (53), (57) and (58) are combined to obtain

$$\begin{cases} V_Q \tau \cdot \boldsymbol{Q}^\eta = 2na\eta^{1+\alpha-2n\beta} u^{n-1} v\left[uY^* \boldsymbol{e}_\eta - \eta^\alpha (1+Y^*)(1+\beta) f f' \boldsymbol{e}_\xi\right] \\ V_Q \tau \cdot \boldsymbol{Q}^\xi = 2na\eta^{-2n\beta} u^{n-1} v\left[-\eta^\alpha (1+Y^*)(1+\beta) f f' \boldsymbol{e}_\eta - u\boldsymbol{e}_\xi\right] \end{cases} \tag{60}$$

in which

$$Y^* = v^{-2n} \tag{61}$$

$$\boldsymbol{e}_\eta = \boldsymbol{Q}_\eta,\ \boldsymbol{e}_\xi = \eta^{-1-\alpha} \boldsymbol{Q}_\xi \tag{62}$$

In order to derive the equilibrium equation, the following relations are needed,

$$\begin{cases} \dfrac{\partial \boldsymbol{e}_\eta}{\partial \eta} = \alpha(1-\alpha)\eta^{\alpha-1}\xi \boldsymbol{e}_\theta, & \dfrac{\partial \boldsymbol{e}_\xi}{\partial \eta} = -\alpha(1+\alpha)\eta^{\alpha-1}\xi \boldsymbol{e}_r \\ \dfrac{\partial \boldsymbol{e}_\eta}{\partial \xi} = (1+\alpha)\eta^\alpha \boldsymbol{e}_\theta, & \dfrac{\partial \boldsymbol{e}_\xi}{\partial \xi} = -(1+\alpha)\eta^\alpha \boldsymbol{e}_r \end{cases} \tag{63}$$

Substituting (60) and (63) into (12) and only taking the dominant terms, we obtain

$$\begin{cases} vu^n = \text{const} = T \\ (1+\beta)\left[u^{-1}(1+Y^*) f' f\right]' - (1+\alpha)(1+Y^*) + 2n\beta Y^* = 0 \end{cases} \tag{64}$$

Eq. (64) can be reduced as

$$\begin{cases} f'' - fg'^2 - \left(\dfrac{u}{2n} + f^2 g'^2\right)\Delta^* = 0 \\ g'' + 2\dfrac{f'}{f}g' + f'g'\Delta^* = 0 \end{cases} \tag{65}$$

where

$$\Delta^* = \frac{\beta}{1+\beta} \cdot \frac{u}{f}\left(1 - \frac{2nY^*}{1+Y^*}\right)\left[\left(1+\frac{1}{2n}\right)u - f'^2\left(1+\frac{1}{n} - \frac{2nY^*}{1+Y^*}\right)\right]^{-1} \tag{66}$$

The displacement conditions for (65) are

$$f(0) = f_0,\ g(0) = \pi - B,\ g'(0) = g'_0 \tag{67}$$

where $B$ is half of the notch angle. Further assuming that, the friction stress on the contact surface is zero, i.e. $\tau^{\xi\eta} = 0$ at $\xi = 0$, then according to Eqs. (11), (57) and (58), it follows that

$$f'(0) = 0 \tag{68}$$

At $\xi = -\infty$, we have the boundary conditions to connect with sector EX,

$$f(-\infty) = +\infty,\ g(-\infty) = 0 \tag{69}$$

Eq. (65) with boundary conditions (67)–(69) can be solved numerically. $f_0$ is a free parameter to indicate the amplitude of the field. $g'_0$ can be adjusted to meet the second condition of (69) when the second of (67) is satisfied. As for the singular exponent $\beta$, it can be determined by matching the sectors EX and SH, see Section 6. For the time being $\beta$ is considered as a known parameter.

It should be noted that, the constant $T$ in Eq. (64) can be determined by $f_0$ and $g_0$,

$$T = -(1+\beta)f_0^{2n+2} \cdot g'^{2n+1}_0 \tag{70}$$

An interesting fact found in the calculation is that for any fixed $\beta$ and $g_0$ when $\xi \to \infty$,

$$Y^* \to \frac{1}{2n-1} \quad \text{or} \quad v \to (2n-1)^{\frac{1}{2n}} \tag{71}$$

This fact is similar to that found by Gao (1998). According to (71) and (66), we can see,when $\xi - \infty$, $\Delta^* \to 0$, therefore (65) is asymptotically expressed as

$$\begin{cases} f'' - fg'^2 = 0 \\ g'' + \dfrac{2f'}{f}g' = 0, \qquad \text{when } \xi - \infty \end{cases} \tag{72}$$

then the asymptotic behavior of $f$ and $g$ is

$$f = -C_f\xi,\ g = -C_g\xi^{-1}, \quad \xi \to -\infty \tag{73}$$

Noting (71), we find that $C_g$ is related with $C_f$ by

$$C_g = \frac{1}{1+\beta}(2n-1)^{\frac{1}{2n}} C_f^{-2} \tag{74}$$

Formula (73) is verified by calculation. The value $C_f$ is proportional to $f_0$.

In order to reduce the calculation work, it is necessary to consider the nature of Eq. (65). Evidently, if $f(\xi)$ and $g(\xi)$ is a set of solution to (65), then the following functions are also a set of solutions,

$$F(\xi) = k^{-1} f\left(k^2\xi\right),\ G(\xi) = g\left(k^2\xi\right), \tag{75}$$

in which $k$ is an arbitrary constant. Therefore, the calculation will be done only for the case of $f_0 = 1$, then $Y^*(0) = [(1+\beta)g_0']^{-2n}$. Actually, the initial value $Y^*(0)$ directly controls the boundary condition (69).

When $Y^*(0) = \frac{1}{2n-1}$ we found a very interesting case that

$$Y^* \equiv \frac{1}{2n-1}, \quad -\infty < \xi < 0 \tag{76}$$

this will result in

$$f'' - fg'^2 = 0,\ g'' + 2\frac{f'}{f}g' = 0 \tag{77}$$

The solution of (77) is

$$f = f_0\left(1 + K^2\xi^2\right)^{1/2},\ g = \frac{\pi}{2} + \mathrm{arctg}(K\xi),\ K = (2n-1)^{1/2n} f^{-2}/(1+\beta) \tag{78}$$

this solution is just for $B = \pi/2$, i.e. the rubber wedge becomes a half plane. So, for the contact problem of a half plane with a rigid wedge, the completely analytical solution (77) is obtained.

Further calculation shows that $Y^*(0) > 1/(2n-1)$ corresponds to $B < \pi/2$; $Y^*(0) < 1/(2n-1)$ corresponds to $B > \pi/2$. So, angle $B$ directly influence the solution.

## 6. Assembly of sectors

As mentioned before, there is no strict boundary between the sectors EX and SH (or SH′). Therefore, the solutions of Eqs. (40) and (41) when $\theta \to \pi/2$ must possess the same meaning as the solutions of Eq. (65) when $\xi \to -\infty$.

According to Eqs. (48), (50) and (73), when $\xi \to -\infty$, we have,

$$R = r^{1-\beta} C_f(\pi - A - \theta),\ \Theta = r^{\alpha} C_g(\pi - A - \theta)^{-1} \tag{79}$$

According to (41) and (44), when $\theta \to \pi - A$, we have

$$h = C_h(\pi - A - \theta),\ m = C_m(\pi - A - \theta)^{-1} \tag{80}$$

Where

$$\begin{cases} C_h = h_0(1-\delta)\varepsilon^{\delta} \\ C_m = \dfrac{1}{1-\delta+\gamma}(2n-1)^{\frac{1}{2n}} C_h^{-2} \end{cases} \tag{81}$$

Substituting Eq. (80) into (18), it follows that

$$R = r^{1-\delta} C_h(\pi - A - \theta),\ \Theta = r^{\gamma} C_m(\pi - A - \theta)^{-1} \tag{82}$$

Comparing Eqs. (79) and (82) it is required that

$$\alpha = \gamma,\ C_f = C_h \tag{83}$$

$$\beta = \delta,\ C_g = C_m \tag{84}$$

Evidently, Eqs. (83) and (84) are consistent with each other because of Eqs. (38), (71), (28) and (59). Eqs. (74), (81), (83) and (84) only permit one free parameter $h_0$ that characterize the amplitude of the field. Thus, the sectors EX and SH are assembled. The assembly of EX and SH′ can be given similarly.

## 7. Numerical result

The numerical calculation is only for sector SH because the analytical solution is obtained for sector EX. Since the varying interval of $\xi(-\infty, 0]$ is not convenient for calculation, the new variable $\zeta$ is introduced such as

$$\zeta = \arctan \xi, \quad -\pi/2 < \zeta \le 0 \tag{85}$$

then Eq. (65) becomes

$$\begin{cases} \ddot{f} - 2\tan\zeta \cdot \dot{f} - f\dot{g}^2\left(\dfrac{u}{2n} + f^2\dot{g}^2\right)\Delta^* = 0 \\ \ddot{g} - 2\tan\zeta \cdot \dot{g} + 2\dot{f}\dot{g}f^{-1} + \dot{f}\dot{g}\Delta^* = 0 \end{cases} \tag{86}$$

where

$$(\cdot) = \frac{\mathrm{d}}{\mathrm{d}\zeta}(\,) \tag{87}$$

in (87), $\Delta^*$ is still given by (66), but $u$ and $Y^*$ are replaced by,

$$\begin{cases} u - \dot{f}^2 - f^2\dot{g}^2 \\ Y^* = \left[(1+\beta)f^2\dot{g}\cos^2\zeta\right]^{-2n} \end{cases} \tag{88}$$

Let

$$Y_0 = (2n-1)Y^*(0) \tag{89}$$

For $n = 2$ and various $A$, $B$, the obtained values of $Y_0$ are listed in Table 1. The curves of $f(\zeta)$ and $g(\zeta)$ for $n = 2$, $A = \pi/12$ and various $B$ are plotted in Fig. 3(a) and (b). Fig. 3 shows that at $\zeta = 0$ $(\xi = 0)$, $f = 1$, $g = \pi - B$; when $\zeta \to -\pi/2$ $(\xi \to -\infty)$, $f \to \infty$, $g \to 0$. It should be noted that, $f = R/r^{1+\beta}$, when $f \to \infty$ the mapping function (17) must be replaced by mapping function (18). The numerical results show that when $A > B$ the calculation is not stable. Besides, when $B$ is small the

Table 1
The values of $Y_0$

| B \ $Y_0$ | A: 0 | $\pi/12$ | $\pi/4$ | $3\pi/8$ |
|---|---|---|---|---|
| $\pi/4$ | 33.6 | 69.6 | | |
| $5\pi/16$ | 11.7 | 14.5 | 171.6 | |
| $3\pi/8$ | 5.20 | 5.77 | 9.81 | |
| $7\pi/16$ | 2.38 | 2.49 | 3.02 | 5.57 |
| $\pi/2$ | 1.0 | 1.0 | 1.0 | 1.0 |
| $9\pi/16$ | 0.342 | 0.327 | 0.275 | 0.185 |
| $5\pi/8$ | 0.0820 | 0.0750 | 0.0533 | 0.0287 |
| $2\pi/3$ | 0.0239 | 0.0213 | 0.0142 | |

convergence is very slow; when $B$ is large the convergence is quick, but the calculation is not stable. For $n = 2$, $A = \pi/12$, when $B < 0.214\pi$ or $B > 0.684\pi$, the calculation cannot be done.

The lack of convergence and the unstability does not mean non-existence of a solution, as frequently happened in the non-linear problems. When $B$ is very small or large, maybe the features of EX sector and SH sector are not so precise, so that the sector division method become not valid.

In order to plot the stress components, let

$$\begin{cases} \tau = \tau^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j & (i, j = r \text{ or } \theta) \\ T^{ij} = r^{2n\delta}(2na)^{-1}\tau^{ij} & (i, j = r \text{ or } \theta) \end{cases} \tag{90}$$

In sector SH, the dominant components of stress are $T^{rr}$ and $T^{\theta\theta}$. However $T^{\theta\theta} = \text{const}$, so only the curves of $T^{rr}$ need to be plotted. From the first of Eq. (60), it follows

$$T^{rr} = Y^* u^n v \tag{91}$$

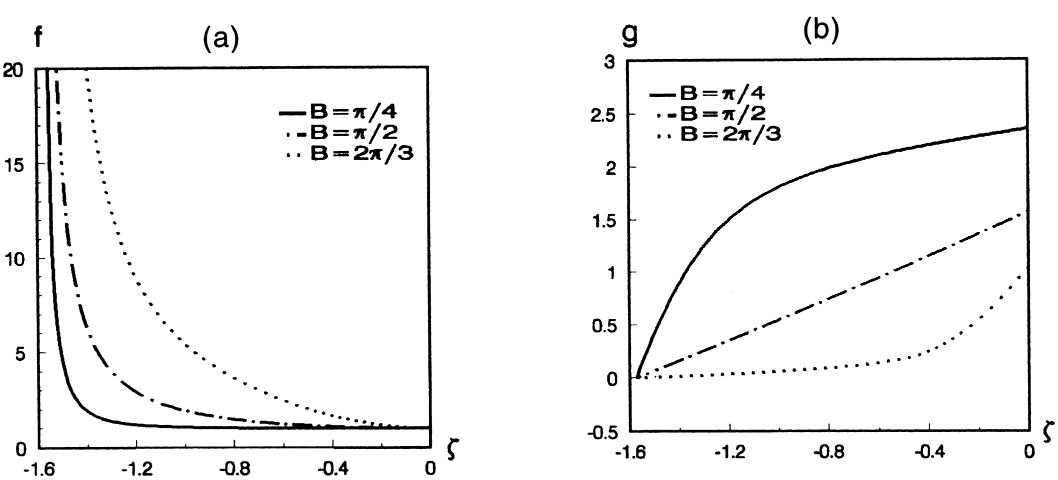


Fig. 3. Curves of $f(\zeta)$ and $g(\zeta)$ for $A = \pi/12, n = 2$. (a) $f(\zeta)$, (b) $g(\zeta)$.

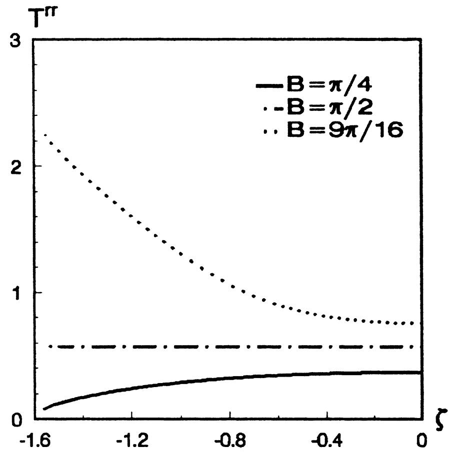


Fig. 4. Curves of $T^{rr}(\zeta)$ in sector SH for $A = \pi/12, n = 2$.

For $f_0 = 1$, $n = 2$, $A = \pi/12$ and various $B$, $T^{rr}(\zeta)$ is plotted in Fig. 4. From Fig. 4 we can see that when $B = \pi/2$, $T^{rr}$ is constant; when $B < \pi/2$, $T^{rr}$ is an increasing function of $\zeta$; when $B > \pi/2$, $T^{rr}$ is a decreasing function of $\zeta$.

The variable in Figs. 3 and 4 is $\zeta$ that can show the convergence precisely, in order to give an intuitive showing of the curves, now we come back to the variable $\xi = -(\pi - A - \theta)r^{-\alpha}$, the curves of $f(\xi)$ and $g(\xi)$ are plotted in Fig. 5. The curves of $T^{rr}(\xi)$ are plotted in Fig. 6. Fig. 5(a) shows that when $\xi \rightarrow -\infty$, function $R/r^{1+\beta}(=f)$ becomes a straight line. Fig. 5(b) shows that when $\xi \rightarrow -\infty$, function $\Theta(=g)$ tends to zero (transfer into EX sector). Fig. 6 shows that when $\xi \rightarrow -\infty$, $T^{rr}$ tends to be a constant; for the case of $B = \pi/2$, $T^{rr}$ keeps constant value everywhere.

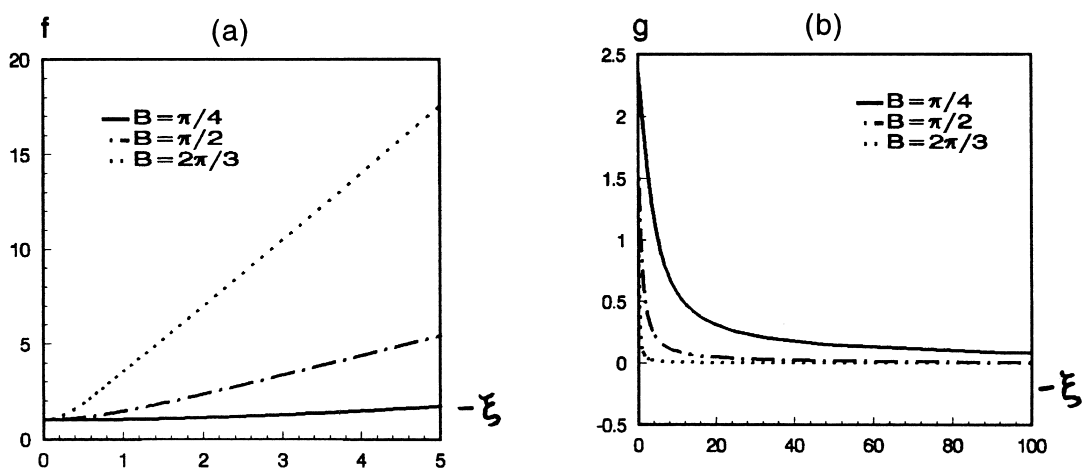


Fig. 5. Curves of $f(\xi), g(\xi)$ for $A = \pi/12, n = 2$. (a) $f(\xi)$, (b) $g(\xi)$.

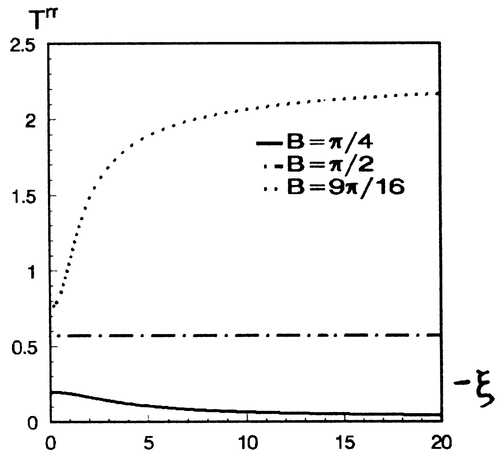


Fig. 6. Curves of $T^{rr}(\xi)$ in sector SH for $A = \pi/12, n = 2$.

In sector EX, from (90) it follows

$$\begin{cases} T^{rr} = p^{n-1}q\left[\dfrac{1}{2n-1}h'^{\,2} - (1-\delta)^2h^2\right], \quad T^{r\theta} = -\dfrac{2n(1-\delta)}{2n-1}p^{n-1}qhh' \\ T^{\theta\theta} = p^{n-1}q\left[\dfrac{(1-\delta)^2}{2n-1}h^2 - h'^{\,2}\right] \end{cases} \tag{92}$$

Using Eqs. (44) and (91), the curves of $T^{rr}$, $T^{r\theta}$ and $T^{\theta\theta}$ are plotted in Fig. 7 for $h_0 = 1$, $n = 2$, $A = \pi/12$. It is shown that at $\theta = \pi - A$, $T^{rr} > 0$, $T^{\theta\theta} < 0$, $T^{r\theta} = 0$. It should be noted that the curves are independent of the angle $B$.

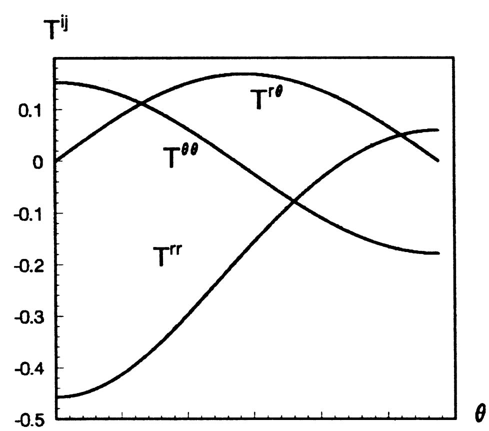


Fig. 7. Curves of normalized stresses in sector EX for $A = \pi/12, n = 2$.

## 8. Conclusions

1. The elastic contact problem of two smooth surfaces can be solved by either linear or nonlinear geometry theory depending on the magnitude of strain.
2. The contact problem with vertex point cannot be solved by linear geometry theory. In the small strain theory, the vertex contact problem is always equivalent to that of a concentrated force acting on the bodies, i.e. the contact area is zero.
3. For the kind of elastic materials discussed in this paper the vertex contact problem was solved. The deformation field contains an expanding sector and two shrinking sectors. The stress and strain possess singularity, $\tau \sim r^{-2n\delta}$, $\boldsymbol{d} \sim r^{-2\delta}$, the singular exponent $\delta$ only depends on the value $n$ and the angle of rigid wedge $A$, but not on the angle of rubber notch (wedge) $B$.
4. For sector EX the analytical solution is similar to that obtained by Gao (1999a) for notch tip field in SH sector but for inverse mapping functions.
5. When $B < 0.214\pi$ or $B > 0.684\pi$, the calculation cannot be done because of very slow convergence or unstability, but this does not mean that the solution does not exist. May be the features of SH sector and EX sector are not that distinct therefore the sector division method is not valid.
6. When $B = \pi/2$, the notch vanishes and becomes a half plane, the completely analytical solution was obtained for both sector EX and SH. This result is consistent with that obtained by Gao (1998), but it is completely different from linear elastic solution.
7. The contact zone is continuous along the wedge franks, and the length of the contact zone depends on the load but that cannot be determined by the asymptotic solution.
8. The solution given in this paper is also valid for the problem of a rubber wedge ($B > \pi/2$) contacting with a rigid wedge.

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